Mathematical Excalibur

Volume 3, Number 1 Jan-Feb, 1997

Olympiad Corner

1997 Chinese Mathematical Olympiad:

Part I (8:00-12:30, January 13, 1997)

Problem 1. Let $x_1, x_2, \ldots, x_{1997}$ be real numbers satisfying the following two conditions:

(1) $-\dfrac{1}{\sqrt{3}} \le x_i \le \sqrt{3}$ $(i = 1, 2, \ldots, 1997)$;

(2) $x_1 + x_2 + \cdots + x_{1997} = -318\sqrt{3}$.

Find the maximum value of

$$x_1^{12} + x_2^{12} + \cdots + x_{1997}^{12}.$$

Problem 2. Let $A_1B_1C_1D_1$ be an arbitrary convex quadrilateral. Let P be a point inside the quadrilateral such that the segments from P to each vertex form acute angles with the two sides through the vertex. Recursively define A_k, B_k, C_k and D_k as the points symmetric to P with respect to the lines $A_{k-1}B_{k-1}$, $B_{k-1}C_{k-1}$, $C_{k-1}D_{k-1}$ and $D_{k-1}A_{k-1}$, respectively $(k = 2, 3, \cdots)$.

(*continued on page 4*)

Acknowledgment: Thanks to Teresa FUNG, EEE Dept, HKUST for her help in typesetting; and Catherine NG, EEE Dept, HKUST for general assistance.

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word are encouraged. The deadline for receiving material for the next issue is Apr. 5, 1997.

For individual subscription for the remaining three issues for the 96-97 academic year, send us three stamped self-addressed envelopes. Send all correspondence to:

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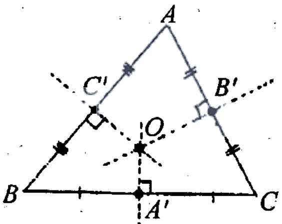
老師不教的幾何（二）

張百康

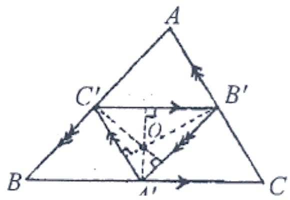
十八世紀時，瑞士出了一位大數學家歐拉 (Euler)。他雖然在二十多歲時已一目失明，但畢生努力從事數學研究，著作豐富。數學上不少定理、公式和方法都是他發現、證明或發明的。我們現在要介紹的是歐拉線 (Euler Line) —— 一條貫穿三角形幾個重要的點的直線。

對一任意的三角形 ABC，通過它的三條邊的中點 (mid-points) A'，B' 和 C' 分別作出這三條邊的垂直平分線 (perpendicular bisectors)。我們知道：這三條垂直平分線相交於同一點，即圖一的點 O。這點 O 就是三角形 ABC 的外接圓心 (circumcentre)，道理相信大家已知道。



(圖一)

另一方面，三角形 $A'B'C'$ 和 ABC 不但相似，而且對應邊平行。這個邊長縮小一半的三角形 $A'B'C'$ 稱為三角形 ABC 的中點三角形 (medial triangle)。它的三條高 (altitudes) 剛好就落在 OA'，OB' 和 OC' 上，因此 O 點也扮演了中點三角形 $A'B'C'$ 的垂心 (orthocentre) 角色 (圖二)。



(圖二)

歐拉發現任何三角形的外接圓心 (O)、重心 (G) 和垂心 (H) 共線，他的證明如下 (圖三)：

由於三角形 ABC 的高 AH 和邊 BC 的垂直平分線 OA' 平行，因此

$$\angle HAG = \angle OA'G$$

並且，AH 和 $A'O$ 分別是相似三角形 ABC 和 $A'B'C'$ 的對應線，所以

$$AH : A'O = BC : B'C' = 2 : 1$$

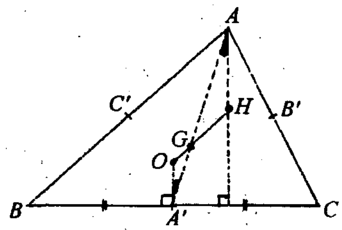
恰巧地，重心 G 也把中線 AA' 分成

$$AG : A'G = 2 : 1$$

因此，三角形 HAG 和 $OA'G$ 相似。由此推知

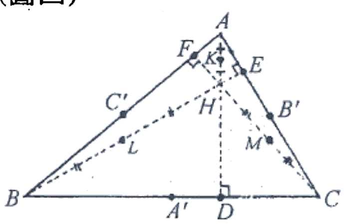
$$\angle HGA = \angle OGA'$$

所以 O、G、H 成一直線，稱為歐拉線，並且 $OG : GH = 1 : 2$。



(圖三)

歐拉線 OH 的中點絕不平凡，它是著名的九點圓 (Nine-point circle) 的圓心。所謂九點圓是指一個通過三角形 ABC 的三邊的中點 A'、B'、C'，三高的垂足 D、E、F 以及三頂點和垂心間的中點 K、L、M 的圓 (圖四)。



(圖四)

有關這九點為甚麼共圓的完整證明是數學家彭賽列 (Poncelet) 於 1821 年首先給出的，他將 A'，B'，C'，K，L，M 六點分成互有重覆四點組合，然後證明每個組合的四點共圓，再利用這三個組合的重覆性證明這三個圓實質上是同一個圓，最後證明 D，E，F 也在這圓上。讓我們看看他的證法：

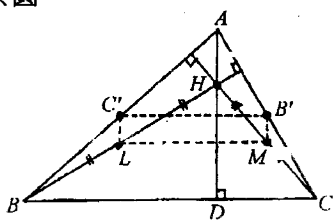
先考慮 B'，C'，L，M 四點 (圖五)。

(*continued on page 2*)

老師不教的幾何 (二)

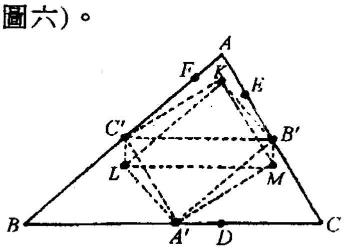
(continued from page 1)

在三角形 ABH 中， C' 和 L 分別是邊 AB 和 HB 的中點，因此 $C'L$ 平行 AH。同理，在三角形 ACH 中，$B'M$ 平行 AH。所以 $C'L$ 平行 $B'M$。再考慮三角形 ABC 和 HBC，利用同樣的中點定理，可知 $B'C'$ 平行 ML 和 CB。由於 AD 垂直 BC，因此 $B'C'LM$ 是個矩形。矩形的頂點當然共圓。



(圖五)

重覆同樣的論証於 $A'C'KM$ 和 $A'B'KL$ 可推證它們也是矩形，因此分別共圓。但這三個矩形兩兩有共同對角線，即外接圓 (circumcircle) 的直徑 (圖六)。

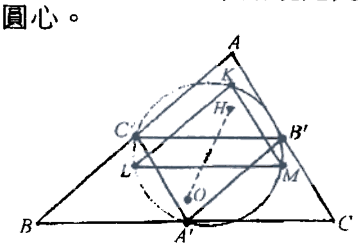


(圖六)

不同的圓不可能有共同直徑，因此 A'，B'，C'，K，L，M 六點共圓。另一方面，$\angle A'DK$ 是直角(圖四)，而 $A'K$ 是前述六點圓的直徑，因此 D 也在此六點圓上。同理，E 和 F 也在此六點圓上，所以九點共圓。

九點圓和歐拉線有甚麼關係？

大家不妨細心比較兩個頂點都在九點圓上的三角形 $A'B'C'$ 和 KLM (圖七)。由於 KA'，LB' 和 MC' 是九點圓的直徑，因此三角形 KLM 繞九點圓的圓心旋轉 180° 可得三角形 $A'B'C'$。三角形 ABC 的歐拉線 OH 兩端恰巧正分別是三角形 $A'B'C'$ 和 KLM 的垂心(可參考圖二及圖四)，因此是全等三角形 $A'B'C'$ 和 KLM 的對應點，它們的中點就是九點圓的圓心。



(圖七)

歐拉線眞不簡單，它一線穿四心，說它是三角形的脊骨一點也不過份。

$\sqrt{2}$ 是無理數的六個證明

香港大學數學系

蕭文強

「如何證明 $\sqrt{2}$ 是無理數呢？」

「那還不容易！設 $\sqrt{2}=m/n$，可當 m 和 n 不全是偶數。由於 $m^2=2n^2$，m 必是偶數，寫作 $2k$，則 $4k^2=2n^2$，$2k^2=n^2$，故 n 亦是偶數，矛盾！」

上述證明，只用到奇偶性質，來源已不可稽考。亞里士多德（ARISTOTLE）在公元前 330 年左右把它（以幾何形式）寫下來，用作反證法的示範，可見在那個時候這回事已是衆所週知了。不過由於這證明是如此簡潔，很多數學史家都相信那不是這回事的發現經過，而是「事後孔明」的解釋。

在這個證明中，2 沒有什麼特別，換了是另一個質數，同樣的思路仍可沿用，只是單憑奇偶性質不足夠，需要用到質因子唯一分解性質。再推廣少許，我們還能夠證明若 P_1、…、P_s 是 s 個不同的質數，則 $\sqrt{P_1\cdots P_s}$ 是無理數。因此，若 H 不是完全平方，則 $\sqrt{H}$ 是無理數。其實，如果我們願意運用質因子唯一分解性質，還有另一個證明辦法，即是數一數 $m^2=Hn^2$ 兩邊中某質因子出現的次數，一奇一偶，矛盾！

讓我們來看第三個證明。設 $\sqrt{H}=m/n$，可當 m 和 n 無公共因子。由於 $m^2=Hn^2=n(Hn)$，n 必須是 1 或 −1，即是說 H 是個完全平方，矛盾！這個證明跟前兩個證明有一點不相同，它能推廣至頗一般的情況，證明了若有理數是代數整數，則它必是整數。（代數整數是指首一整數系數多項式方程 $x^N+c_{N-1}x^{N-1}+\cdots+c_1x+c_0=0$ 的根，例如 $\sqrt{H}$ 是 $x^2-H=0$ 的根。請讀者試自行證明這回事吧。）

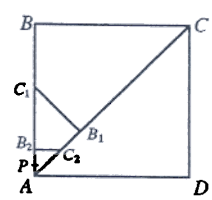
現在再看一個十分簡捷的證明：若 $\sqrt{2}$ 是有理數，取最小正整數 k 使 $k\sqrt{2}$ 是整數，則 $m=k\sqrt{2}-k=k(\sqrt{2}-1)$ 是一個較 k 更小的正整數，但 $m\sqrt{2}=2k-k\sqrt{2}$ 仍是整數，這與 k 的選取矛盾！（把 2 換作一個非完全平方 H，類似的證明適用。）

上述證明是數論專家埃斯特曼（THEODOR ESTERMANN）在 1975 年一則短文的內容，巧妙簡捷，兼而有之。後來有人讚曰：「如同所有精采念頭，一經指出即明顯不過，但這個精采念頭卻要等到畢達哥拉斯 (PYTHAGORAS)二千多年後才給指出來！」如果我們試圖追尋如何選取 m 的線索，自然會問到它的幾何詮釋，這個幾何詮釋，說不定正是二千多年前希臘數學家發現正方形的對角線和邊是不可公度量的經過呢！不可公度量，是指不存在一公共度量，使對角線和邊各自是該公共度量的若干整數倍，也就是說，$\sqrt{2}$ 不是有理數。（以下敘述，取材於 H. EVES 的著作 "AN INTRODUCTION TO THE HISTORY OF MATHEMATICS" 的第 3 章，3rd edition，1969。）在下圖中設 AP 是正方形的對角線 AC 和邊 AB 的公共度量，即有 $AC=jAP$ 和 $AB=kAP$。構作 B_1C_1 使 B_1C_1 垂直於 AC，也使 $CB=CB_1$。不難知道 $BC_1=B_1C_1=AB_1$，因此

$$AC_1=AB-AB_1=AB-(AC-AB)=2AB-AC=(2k-j)AP\ ,$$

$$AB_1=AC-AB=(j-k)AP\ 。$$

注意：AC_1 和 AB_1 是一個較小的正方形的對角線和邊，那個較小的正方形的邊 AB_1 小於原正方形的邊 AB 的一半。按此步驟重複下去，必得到一個足夠小的正方形，它的邊 AB_t 小於 AP，但 AB_t 卻仍然是 AP 的若干整數倍，豈非矛盾！（有些數學史家認爲古代希臘數學家曾企圖以此方法研究不可公度量理論，相當於企圖發展今天稱作連分數展開式的研究。可惜當時的數學家無功而退，只遺留下來蛛絲馬跡，在古希臘數學名著《歐幾里得原本》(EUCLID'S ELEMENTS) 的章節間依稀可見！）



請注意：$AC_1/AB_1=(2k-j)/(j-k)$，而 $m=j-k$ 正是埃斯特曼的短小精悍證明中的 m。因爲 $AC_1/AB_1=\sqrt{2}$，便有 $(2k-j)/m=\sqrt{2}$，即是 $m\sqrt{2}=2k-j$ 是整數了。當我們了解埃斯特曼證明的背後的幾何詮釋，我們可以把它重寫成第六個證明：若 $\sqrt{2}=j/k$ 是最簡的分數式，則有 $\sqrt{2}=(2k-j)/(j-k)$（這是因爲 $j\sqrt{2}-k\sqrt{2}=2k-j$），但 $k<j<2k$（因爲 $1<\sqrt{2}<2$），故 $2k-j<j$ 和 $j-k<k$，這與 j 和 k 的選取矛盾！

請讀者想一想，上面討論的六個證明，眞的是六個不同的證明嗎？還是六個相同的證明呢？

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is Apr. 5, 1997.

Problem 51. Is there a positive integer n such that $\sqrt{n-1}+\sqrt{n+1}$ is a rational number?

Problem 52. Let a, b, c be distinct real numbers such that $a^3 = 3(b^2+c^2) - 25$, $b^3 = 3(c^2+a^2) - 25$, $c^3 = 3(a^2+b^2) - 25$. Find the value of abc.

Problem 53. For ΔABC, define A' on BC so that $AB + BA' = AC + CA'$ and similarly define B' on CA and C' on AB. Show that AA', BB', CC' are concurrent. (The point of concurrency is called the *Nagel point* of ΔABC.)

Problem 54. Let R be the set of real numbers. Find all functions $f : R \to R$ such that

$$f(f(x+y)) = f(x+y) + f(x)f(y) - xy$$

for all $x, y \in R$. (Source: 1995 Byelorussian Mathematical Olympiad (Final Round))

Problem 55. In the beginning, 65 beetles are placed at different squares of a 9×9 square board. In each move, every beetle creeps to a horizontal or vertical adjacent square. If no beetle makes either two horizontal moves or two vertical moves in succession, show that after some moves, there will be at least two beetles in the same square. (Source: 1995 Byelorussian Mathematical Olympiad (Final Round))

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Solutions

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Problem 46. For what integer a does $x^2 - x + a$ divide $x^{13} + x + 90$? (Source: 1963 Putnam Exam.)

Solution: CHEUNG Tak Fai (Valtorta College, Form 6) and **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4).

Suppose

$$x^{13} + x + 90 = (x^2 - x + a)q(x),$$

where $q(x)$ is a polynomial with integer coefficients. Taking $x = -1, 0, 1$, we get

$$88 = (2+a)q(-1),$$
$$90 = aq(0)$$
and
$$92 = aq(1).$$

Since a divides 90, 92 and $a+2$ divides 88, a can only be 2 or -1. Now $x^2 - x - 1$ has a positive root, but $x^{13} + x + 90$ cannot have a positive root. So a can only be 2. We can check by long division that $x^2 - x + 2$ divides $x^{13} + x + 90$ or observe that if w is any of the two roots of $x^2 - x + 2$, then $w^2 = w - 2$, $w^4 = -3w + 2$, $w^8 = -3w - 14$, $w^{12} = 45w - 46$ and $w^{13} + w + 90 = 0$.

Other commended solvers: **CHAN Ming Chiu** (La Salle College, Form 6), **CHAN Wing Sum** (HKUST) and **William CHEUNG Pok-man** (S.T.F.A. Leung Kau Kui College, Form 6).

Problem 47. If x, y, z are real numbers such that $x^2 + y^2 + z^2 = 2$, then show that $x + y + z \le xyz + 2$.

Solution: CHAN Ming Chiu (La Salle College, Form 6).

If one of x, y, z is nonpositive, say z, then

$$2 + xyz - x - y - z = (2-x-y) - z(1-xy) \ge 0$$

because

$$x + y \le \sqrt{2(x^2+y^2)} \le 2$$

and

$$xy \le (x^2 + y^2)/2 \le 1$$

So we may assume x, y, z are positive, say $0 < x \le y \le z$. If $z \le 1$, then

$$2 + xyz - x - y - z = (1-x)(1-y) + (1-z)(1-xy) \ge 0.$$

If $z > 1$, then

$$(x+y) + z \le \sqrt{2\left((x+y)^2 + z^2\right)} = 2\sqrt{xy+1} \le xy + 2 \le xyz + 2.$$

Comments: This was an unused problem in the 1987 IMO and later appeared as a problem on the 1991 Polish Mathematical Olympiad.

Problem 48. Squares $ABDE$ and $BCFG$ are drawn outside of triangle ABC. Prove that triangle ABC is isosceles if DG is parallel to AC.

Solution: Henry NG Ka Man (STFA Leung Kau Kui College, Form 6), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4) and **YUNG Fai** (CUHK).

From B, draw a perpendicular line to AC (and hence also perpendicular to DG.) Let it intersect AC at X and DG at Y. Since $\angle ABX = 90° - \angle DBY = \angle BDY$ and $AB = BD$, the right triangles ABX and BDY are congruent and $AX = BY$. Similarly, the right triangles CBX and BGY are congruent and $BY = CX$. So $AX = CX$, which implies $AB = CB$.

Comments: This was a problem on the 1988 Leningrad Mathematical Olympiad. Most solvers gave solutions using pure geometry or a bit of trigonometry. The editor will like to point out there is also a simple vector solution. Set the origin O at the midpoint of AC. Let $\vec{OC} = m$, $\vec{OB} = n$ and k be the unit vector perpendicular to the plane. Then $\vec{AB} = n + m$, $\vec{CB} = n - m$, $\vec{BD} = -(n + m) \times k$, $\vec{BG} = (n - m) \times k$ and $\vec{DG} = \vec{BG} - \vec{BD} = 2n \times k$. If DG is parallel to AC, then $n \times k$ is a multiple of m and so $m = \vec{OC}$ and $n = \vec{OB}$ are perpendicular. Therefore, triangle ABC is isosceles.

Other commended solvers: **CHAN Wing Chiu** (La Salle College, Form 4), **Calvin CHEUNG Cheuk Lun** (S.T.F.A. Leung Kau Kui College, Form 5), **William CHEUNG Pok-man** (S.T.F.A. Leung Kau Kui College, Form 6), **Yves CHEUNG Yui Ho** (S.T.F.A. Leung Kau Kui College, Form 5), **CHING Wai Hung** (S.T.F.A. Leung Kau Kui College, Form 5), **Alan LEUNG Wing Lun** (STFA Leung Kau Kui College, Form 5), **OR Fook Sing** & **WAN Tsz Kit** (Valtorta College, Form 6), **TSANG Sai Wing** (Valtorta College, Form 6), **WONG Hau Lun** (STFA Leung Kau Kui College, Form 5), **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4).

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

Problem 49. Let u_1, u_2, u_3, … be a sequence of integers such that $u_1 = 29$, $u_2 = 45$ and $u_{n+2} = u_{n+1}^2 - u_n$ for $n = 1, 2, 3, \ldots$. Show that 1996 divides infinitely many terms of this sequence. (Source: 1986 Canadian Mathematical Olympiad with modification)

Solution: William CHEUNG Pok-man (STFA Leung Kau Kui College, Form 6) and **YUNG Fai** (CUHK).

Let U_n be the remainder of u_n upon division by 1996, i.e.,

$$U_n \equiv u_n \pmod{1996}.$$

Consider the sequence of pairs (U_n, U_{n+1}). There are at most 1996^2 distinct pairs. So let $(U_p, U_{p+1}) = (U_q, U_{q+1})$ be the first repetition with $p < q$. If $p > 1$, then the recurrence relation implies $(U_{p-1}, U_p) = (U_{q-1}, U_q)$ resulting in an earlier repetition. So $p = 1$ and the sequence of pairs (U_n, U_{n+1}) is periodic with period $q - 1$. Since $u_3 = 1996$, we have $0 = U_3 = U_{3+k(q-1)}$ and so 1996 divides $u_{3+k(q-1)}$ for every positive integer k.

Other commended solvers: **CHAN Ming Chiu** (La Salle College, Form 6), **CHAN Wing Sum** (HKUST) and **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4).

Problem 50. Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and next number on the circle in a given direction (that is, the numbers a, b, c, d are replaced by $a - b$, $b - c$, $c - d$, $d - a$). Is it possible after 1996 such steps to have numbers a, b, c, d such that the numbers $|bc - ad|$, $|ac - bd|$, $|ab - cd|$ are primes? (Source: unused problem in the 1996 IMO.)

Solution 1: Henry NG Ka Man (STFA Leung Kau Kui College, Form 6) and **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4).

If the initial numbers are $a = w$, $b = x$, $c = y$, $d = z$, then after 4 steps, the numbers will be

$$a = 2(w - 2x + 3y - 2z),$$
$$b = 2(x - 2y + 3z - 2w),$$
$$c = 2(y - 2z + 3w - 2x),$$
$$d = 2(z - 2w + 3y - 2z).$$

From that point on, a, b, c, d will always be even, so $|bc - ad|$, $|ac - bd|$, $|ab - cd|$ will always be divisible by 4.

Solution 2: Official Solution.

After $n \geq 1$ steps, the sum of the integers will be 0. So $d = -a - b - c$. Then

$$\begin{aligned} bc - ad &= bc + a(a + b + c) \\ &= (a + b)(a + c). \end{aligned}$$

Similarly,

$$ac - bd = (a + b)(b + c)$$

and

$$ab - cd = (a + c)(b + c).$$

Finally $|bc - ad|$, $|ac - bd|$, $|ab - cd|$ cannot all be prime because their product is the square of $(a+b)(a+c)(b+c)$.

Other commended solvers: **Calvin CHEUNG Cheuk Lun** (S.T.F.A. Leung Kau Kui College, Form 5) and **William CHEUNG Pok-man** (STFA Leung Kau Kui College, Form 6).

Olympiad Corner

(*continued from page 1*)

Consider the sequence of quadrilaterals

$$A_jB_jC_jD_j \quad (j = 1, 2, \cdots).$$

(1) Determine which of the first 12 quadrilaterals are similar to the 1997th quadrilateral.

(2) If the 1997th quadrilateral is cyclic, determine which of the first 12 quadrilaterals are cyclic.

Problem 3. Prove that there are infinitely many natural numbers n such that

$$1, 2, \cdots, 3n$$

can be put into an array

$$\begin{matrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \end{matrix}$$

satisfying the following two conditions:

(1) $a_1+b_1+c_1 = a_2+b_2+c_2 = \cdots = a_n+b_n+c_n$ and the sum is a multiple of 6;

(2) $a_1+a_2+\cdots+a_n = b_1+b_2+\cdots+b_n = c_1+c_2+\cdots+c_n$ and the sum is a multiple of 6.

Part II (8:00-12:30, January 14, 1997)

Problem 4. Let quadrilateral $ABCD$ be inscribed in a circle. Suppose lines AB and DC intersect at P and lines AD and BC intersect at Q. From Q, construct the two tangents QE and QF to the circle where E and F are the points of tangency. Prove that the three points P, E, F are collinear.

Problem 5. Let $A = \{1, 2, 3, \cdots, 17\}$. For a mapping $f: A \to A$, denote

$$f^{[1]}(x) = f(x),$$
$$f^{[k+1]}(x) = f\left(f^{[k]}(x)\right) \quad (k = 1, 2, 3,$$

Consider one-to-one mappings f from A to A satisfying the condition: there exists a natural number M such that

(1) for $m < M$, $1 \leq i \leq 16$,
$$f^{[m]}(i+1) - f^{[m]}(i) \not\equiv \pm 1 \pmod{17},$$
$$f^{[m]}(1) - f^{[m]}(17) \not\equiv \pm 1 \pmod{17};$$

(2) for $1 \leq i \leq 16$,
$$f^{[M]}(i+1) - f^{[M]}(i) \equiv 1 \text{ or } -1 \pmod{17},$$
$$f^{[M]}(1) - f^{[M]}(17) \equiv 1 \text{ or } -1 \pmod{17}.$$

For all mappings f satisfying the above condition, determine the largest possible value of the corresponding M's.

Problem 6. Consider a sequence of nonnegative real numbers a_1, a_2, … satisfying the condition

$$a_{n+m} \leq a_n + a_m, \quad m, n \in N.$$

Prove that for any $n \geq m$,

$$a_n \leq ma_1 + \left(\frac{n}{m} - 1\right)a_m.$$

大拇指與小拇指