Mathematical Excalibur

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Olympiad Corner

The 32nd Austrian Mathematical Olympiad 2001.

Problem 1. Prove that

$$\frac{1}{25}\sum_{k=0}^{2001}\left[\frac{2^k}{25}\right]$$

is an integer. ([x] denotes the largest integer less than or equal to x.)

Problem 2. Determine all triples of positive real numbers x, y and z such that both $x + y + z = 6$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 2-\frac{4}{xyz}$ hold.

Problem 3. We are given a triangle ABC and its circumcircle with mid-point U and radius r. The tangent c' of the circle with mid-point U and radius $2r$ is determined such that C lies between $c = AB$ and c', and a' and b' are defined analogously, yielding the triangle $A'B'C'$. Prove that the lines joining the mid-points of corresponding sides of ΔABC and $\Delta A'B'C'$ pass through a common point.

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 15, 2002***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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對 數 表 的 構 造

李 永 隆

在現今計算工具發達的年代，要找出如 *ln*2 這個對數值只需一指之勞。但是大家有沒有想過，在以前計算機尚未出現的時候，那些厚厚成書的對數表是如何精確地構造出來的？ 當然，在歷史上曾出現很多不同的構造方法，各有其所長，但亦各有其所限。下面我們將會討論一個比較有系統的方法，它只需要用上一些基本的微積分技巧，就能夠有效地構造對數表到任意的精確度。

首先注意，$\ln(xy) = \ln x + \ln y$, 所以我們只需求得所有質數 p 的對數值便可以由此算得其他正整數的對數值。 由 $\ln(1 + t)$ 的微分運算和幾何級數公式直接可得

$$\frac{d}{dt}\ln(1+t)=\frac{1}{1+t}=1-t+t^2-t^3+\cdots+(-1)^{n-1}t^{n-1}+\frac{(-1)^n t^n}{1+t}$$

運用微積分基本定理（亦即微分和積分是兩種互逆的運算），即得下式：

$$\ln(1+x)=\int_0^x\frac{1}{1+t}dt=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+(-1)^{n-1}\frac{x^n}{n}+\int_0^x\frac{(-1)^n t^n}{1+t}dt$$

能夠對於所有正整數 n 皆成立。 現在我們去估計上式中的積分餘項的大小。 設 $|x| < 1$，則有：

$$\left|\int_0^x\frac{(-1)^n t^n}{1+t}dt\right|\le\left|\int_0^x\left|\frac{t^n}{1+t}\right|dt\right|\le\left|\int_0^x\frac{t^n}{1-|x|}dt\right|=\frac{|x|^{n+1}}{(n+1)(1-|x|)}$$

由此可見，這個餘項的絕對值會隨著 n 的增大而趨向 0。換句話說，只要 n 選得足夠大， $\ln(1+x)$ 和 $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+(-1)^{n-1}\frac{x^n}{n}$ 之間的誤差就可以小到任意小，所以我們不妨改用下式表達這個情況：

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\ (|x|<1)$$

總之 n 的選取總是可以讓我們忽略兩者的誤差。 把上式中的 x 代以-x 然後將兩式相減，便可以得到下面的公式：

$$\ln\left(\frac{1+x}{1-x}\right)=\ln(1+x)-\ln(1-x)=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right)\qquad(*)$$

可惜的是若直接代入 $x=\frac{p-1}{p+1}$ 使得 $\frac{1+x}{1-x}=p$ 時，$(*)$−式並不能有效地計算 $\ln p$。例如取 $p=29$，則 $x=\frac{29-1}{29+1}=\frac{14}{15}$，在此時即使計算了 100 項至 $\frac{2x^{199}}{199}\approx 1.1\times10^{-8}$ ，$\ln p$ 的數值還未必能準確至第 8 個小數位（嚴格來說，應該用 $(*)$−式的積分餘項來做誤差估計，不過在這裏我們只是想大約知道其大小）；又例如取 $p=113$，則 $x=\frac{56}{57}$ 而 $\frac{2x^{199}}{199}\approx 3\times10^{-4}$ ，$\ln p$ 的準確度則更差。 但是我們可以取 $x=\frac{1}{2p^2-1}$ ，則有

$$\ln\left(\frac{1+x}{1-x}\right)=\ln\frac{2p^2-1+1}{2p^2-1-1}=\ln\frac{p^2}{(p+1)(p-1)}=2\ln p-\ln(p+1)(p-1)$$

而當質數 $p > 2$ 時，$(p + 1)$ 和 $(p - 1)$ 的質因數都必定小於 p，所以如果我們已算得小於 p 的質數的對數值，就可以用上式來計算 $\ln p$ 的值：

$$2\ln p = \ln\left(\frac{1+x}{1-x}\right)+\ln(p+1)+\ln(p-1)$$

而未知的 $\ln\left(\frac{1+x}{1-x}\right)$ 是能夠有效計算的，因為現在所選的 x 的絕對值很小。例如當 $p = 29$ 時，$x = \frac{1}{2\cdot 29^2-1}$ $= \frac{1}{1681}$，所以只需計算到 $\frac{2x^5}{5} \approx 3 \times 10^{-17}$，便能夠準確至十多個小數位了。

經過上面的討論，假設現在我們想構造一個 8 位對數表，則可以依次序地求 2, 3, 5, 7, 11, 13, … 的對數值，而後面質數的對數值都可以用前面的質數的對數值來求得。由此可見，在開始時的 ln 2 是需要算得準確一些：

$$\ln 2 = \ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)$$
$$\approx 2\left(\frac{1}{3}+\frac{\left(\frac{1}{3}\right)^3}{3}+\frac{\left(\frac{1}{3}\right)^5}{5}+\dots+\frac{\left(\frac{1}{3}\right)^{21}}{21}\right)$$
$$= 0.6931471805589\ldots$$

這個和確實數值

$$\ln 2 = 0.693147180559945\ \ldots$$

相比其精確度已到達第 11 位小數。

接著便是要計算 ln 3。取 $x = \frac{1}{2\cdot 3^2-1}$ $= \frac{1}{17}$，則有

$$\ln\left(\frac{1+\frac{1}{17}}{1-\frac{1}{17}}\right)\approx 2\left(\frac{1}{17}+\frac{\left(\frac{1}{17}\right)^3}{3}+\frac{\left(\frac{1}{17}\right)^5}{5}+\frac{\left(\frac{1}{17}\right)^7}{7}\right)$$
$$= 0.117783035654504\ldots$$

注意 $\frac{2\left(\frac{1}{17}\right)^9}{9} \approx 1.9\times 10^{-12}$，在 8 位的精確度之下大可以不用考慮。 所以

$$\ln 3 \approx \frac{1}{2}\left(0.11778303565\ldots+\ln 4+\ln 2\right)$$
$$= 1.098612288635\ldots$$

（續於第四頁）

Pell's Equation (II)

Kin Y. Li

For a fixed nonzero integer N, as the case $N = -1$ shows, the generalized equation $x^2 - dy^2 = N$ may not have a solution. If it has a least positive solution (x_1, y_1), then $x^2 - dy^2 = N$ has infinitely many positive solutons given by (x_n, y_n), where

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})(a + b\sqrt{d})^{n-1}$$

and (a, b) is the least positive solution of $x^2 - dy^2 = 1$. However, in general these do not give all positive solutions of $x^2 - dy^2 = N$ as the following example will show.

Example 9. Consider the equation $x^2 - 23y^2 = -7$. It has $(x_1, y_1) = (4, 1)$ as the least positive solution. The next two solutions are (19, 4) and (211, 44). Now the least positive solution of $x^2 - 23y^2 = 1$ is $(a, b) = (24, 5)$. Since $(4 + \sqrt{23})(24 + 5\sqrt{23}) = 211 + 44\sqrt{23}$, the solution (19, 4) is skipped by the formula above.

In case $x^2 - dy^2 = N$ has positive solutions, how do we get them all? A solution (x, y) of $x^2 - dy^2 = N$ is called *primitive* if x and y (and N) are relatively prime. For $0 \le s < |N|$, we say the solution belong to class C_s if $x \equiv sy \pmod{|N|}$. As x, y are relatively prime to N, so is s. Hence, there are at most $\phi(|N|)$ classes of primitive solutions, where $\phi(k)$ is *Euler's ϕ-function* denoting the number of positive integers $m \le k$ that are relatively prime to k. Also, for such s, $(s^2 - d)y^2 \equiv x^2 - dy^2 \equiv 0 \pmod{|N|}$ and y, N relatively prime imply $s^2 \equiv d \pmod{|N|}$.

Theorem. *Let (a_1, b_1) be a C_s primitive solutions of $x^2 - dy^2 = N$. A pair (a_2, b_2) is also a C_s primitive solution of $x^2 - dy^2 = N$ if and only if $a_2 + b_2\sqrt{d} = (a_2 - b_2\sqrt{d})/(a_1 - b_1\sqrt{d})$. Multiplying these two equations, we get $u^2 - dv^2 = N/N = 1$.*

To see u, v are integers, note $a_1a_2 - db_1b_2 \equiv (s^2 - d)b_1b_2 \equiv 0 \pmod{|N|}$, which implies u is an integer. Since $a_1b_2 - b_1a_2 \equiv sb_1b_2 - b_1sb_2 = 0 \pmod{|N|}$, v is also an integer.

For the converse, multiplying the equation with its conjugate shows (a_2, b_2) solves $x^2 - dy^2 = N$. From $a_2 = ua_1 + dvb_1$ and $b_2 = ub_1 + va_1$, we get $a_2 = ua_2 - dvb_2$ and $b_1 = ub_2 - va_2$. Hence, common divisors of a_2, b_2 are also common divisors a_1, b_1. So a_2, b_2 are relatively prime. Finally, $a_2 - sb_2 \equiv (usb_1 + dvb_1) - s(ub_1 + vsb_1) = (d - s^2)vb_1 \equiv 0 \pmod{|N|}$ concludes the proof.

Thus, all primitive solutions of $x^2 - dy^2 = N$ can be obtained by finding a solution (if any) in each class, then multiply them by solutions of $x^2 - dy^2 = 1$. For the nonprimitive solutions, we can factor the common divisors of a and b to reduce N.

Example 10. (1995 IMO proposal by USA leader T. Andreescu) Find the smallest positive integer n such that $19n + 1$ and $95n + 1$ are both integer squares.

Solution. Let $95n + 1 = x^2$ and $19n + 1 = y^2$, then $x^2 - 5y^2 = -4$. Now $\phi(4) = 2$ and (1, 1), (11, 5) are C_1, C_3 primitive solutions, respectively. As (9, 4) is the least positive solution of $x^2 - 5y^2 = 1$ and $9 + 4\sqrt{5} = (2 + \sqrt{5})^2$, so the primitive positive solutions are pairs (x, y), where $x + y\sqrt{5} = (1+\sqrt{5})(2+\sqrt{5})^{2n-2}$ or $(11 + 5\sqrt{5})(2+\sqrt{5})^{2n-2}$.

Since the common divisors of x, y divide 4, the nonprimitive positive solutions are the cases x and y are even. This reduces to considering $u^2 - 5v^2 = -1$, where we take $u = x/2$ and $v = y/2$. The least positive solution for $u^2 - 5v^2 = -1$ is (2, 1). So $x + y\sqrt{5} = 2(u + v\sqrt{5}) = 2(2 + \sqrt{5})^{2n-1}$.

In attempt to combine these solutions, we look at the powers of $1+\sqrt{5}$ coming from the least positive solutions (1, 1).

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon*. The deadline for submitting solutions is ***May 15, 2002***.

Problem 146. Is it possible to partition a square into a number of congruent right triangles each containing an 30° angle? (*Source: 1994 Russian Math Olympiad, 3rd Round*)

Problem 147. Factor x^8+4x^2+4 into two nonconstant polynomials with integer coefficients.

Problem 148. Find all distinct prime numbers p, q, r, s such that their sum is also prime and both p^2+qs, p^2+qr are perfect square numbers. (*Source: 1994 Russian Math Olympiad, 4th Round*)

Problem 149. In a 2000×2000 table, every square is filled with *a* 1 or –1. It is known that the sum of these numbers is nonnegative. Prove that there are 1000 columns and 1000 rows such that the sum of the numbers in these intersection squares is at least 1000. (*Source: 1994 Russian Math Olympiad, 5th Round*)

Problem 150. Prove that in a convex n-sided polygon, no more than n diagonals can pairwise intersect. For what n, can there be n pairwise intersecting diagonals? (Here intersection points may be vertices.) (*Source: 1962 Hungarian Math Olympiad*)

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Solutions

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Problem 141. Ninety-eight points are given on a circle. Maria and José take turns drawing a segment between two of the points which have not yet been joined by a segment. The game ends when each point has been used as the endpoint of a segment at least once. The winner is the player who draws the last segment. If José goes first, who has a winning strategy? (*Source: 1998 Iberoamerican Math Olympiad*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), 何思銳 (大角嘴天主教小學, Primary 5), **LAM Sze Yui** (Carmel Divine Grace Foundation Secondary School, Form 4), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), LEUNG Chi Man (Cheung Sha Wan Catholic Secondary School, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **POON Yiu Keung** (HKUST, Math Major, Year 1), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **Ricky TANG** (La Salle College, Form 4), **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6) and **WONG Wing Hong** (La Salle College, Form 4).

José has the following winning strategy. He will let Maria be the first person to use the ninety-sixth unused point. Since there are $C_2^{95}=4465$ segments joining pairs of the first ninety-five points, if Maria does not use the ninety-sixth point, José does not have to use it either. Once Maria starts using the ninety-sixth point, José can win by joining the ninety-seventh and ninety-eighth points.

Problem 142. $ABCD$ is a quadrilateral with $AB \parallel CD$. P and Q are on sides AD and BC respectively such that $\angle APB = \angle CPD$ and $\angle AQB = \angle CQD$. Prove that P and Q are equal distance from the intersection point of the diagonals of the quadrilateral. (*Source: 1994 Russian Math Olympiad, Final Round*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7) and **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6).

Let O be the intersection point of the diagonals. Since $\Delta AOB, \Delta COD$ are similar, $AO:CO = AB:CD = BO:DO$. By sine law,

$$\frac{AB}{BP}=\frac{\sin\angle APB}{\sin\angle BAP}=\frac{\sin\angle CPD}{\sin\angle CDP}=\frac{CD}{CP}.$$

So $AB:CD = BP:CP$. Let S be on BC so that $SP\perp AD$ and R be on AD so that $RQ\perp BC$. Then SP bisects $\angle BPC$, $BS:CS = BP:CP = AB:CD = AO:CO$. This implies $OS \parallel AB$. Then $AB:OS = CA:CO$. Similarly, $AB:RO = DB:DO$. However,

$$\frac{CA}{CO}=1+\frac{AO}{CO}=1+\frac{BO}{DO}=\frac{DB}{DO}.$$

So $OS = RO$. Since O is the midpoint of RS and ΔSPR, ΔRQS are right triangles, $PO = OS = QO$.

Other commended solvers: **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6).

Problem 143. Solve the equation cos cos cos cos x = sin sin sin sin x. (*Source: 1994 Russian Math Olympiad, 4th Round*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7)**.**

Let $f(x)$ = sin sin x and $g(x)$ = cos cos x. Now

$$g(x)-f(x)=\sin\left(\frac{\pi}{2}-\cos x\right)-\sin\sin x$$
$$=2\cos\left(\frac{\pi}{4}-\frac{\cos x}{2}+\frac{\sin x}{2}\right)$$
$$\times\sin\left(\frac{\pi}{4}-\frac{\cos x}{2}-\frac{\sin x}{2}\right)$$

and

$$\left|\frac{\cos x\pm\sin x}{2}\right|=\frac{\sqrt{2}\left|\sin(x\pm\pi/4)\right|}{2}<\frac{\pi}{4}.$$

So $g(x)-f(x)>0$ (hence $g(x) > f(x)$) for all x. Since sin x, $f(x)$, $g(x) \in [-1,1]$ $\subset[-\pi/2,\ \pi/2]$ and sin x is strictly increasing in $[-\pi/2,\pi/2]$, so $f(x)$ is strictly increasing in $[-\pi/2,\ \pi/2]$ and

$$f\left(f(x)\right)<f\left(g(x)\right)<g\left(g(x)\right)$$

for all x. Therefore, the equation has no solution.

*Other commended solvers***:** **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **OR Kin** (HKUST, Year 1) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6).

Problem 144. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) Find all (non-degenerate) triangles ABC with consecutive integer sides a, b, c and such that $C=2A$.

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **KWOK Tik Chun** (STFA Leung Kau Kui College, Form 4), **LAM Wai Pui Billy** (STFA Leung

Kau Kui College, Form 4), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **POON Ming Fung** (STFA Leung Kau Kui College, Form 4), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7), **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6) and **YEUNG Wing Fung** (STFA Leung Kau Kui College).

Let $a=BC$, $b=CA$, $c=AB$. By sine and cosine laws,

$$\frac{c}{a}=\frac{\sin C}{\sin A}=2\cos A=\frac{b^2+c^2-a^2}{bc}.$$

This gives $bc^2=ab^2+ac^2-a^3$. Factoring, we get $(a-b)(c^2-a^2-ab)=0$. Since the sides are consecutive integers and $C>A$ implies $c>a$, we have $(a, b, c) = (n, n-1, n+1)$, $(n-1, n+1, n)$ or $(n-1, n, n+1)$ for some positive integer $n>1$. Putting these into $c^2-a^2-ab=0$, the first case leads to $-n^2+3n+1=0$, which has no integer solution. The second case leads to $2n-n^2=0$, which yields a degenerate triangle with sides 1, 2, 3. The last case leads to $5n-n^2=0$, which gives $(a, b, c) = (4, 5, 6)$.

Other commended solvers: **CHENG Ka Wai** (STFA Leung Kau Kui College, Form 4), **Clark CHONG Fan Fei** (Queen's College, Form 5), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7) and **WONG Wing Hong** (La Salle College, Form 4).

Problem 145. Determine all natural numbers $k>1$ such that, for some distinct natural numbers m and n, the numbers k^m+1 and k^n+1 can be obtained from each other by reversing the order of the digits in their decimal representations. (*Source: 1992 CIS Math Olympiad*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **Ricky TANG** (La Salle College, Form 4) and **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6).

Without loss of generality, suppose such numbers exist and $n>m$. By the required property, both numbers are not power of 10. So k^n and k^m have the same number of digits. Then $10>\frac{k^n}{k^m}=k^{n-m}\geq k$. Since every number and the sum of its digits are congruent (mod 9), we get $k^n+1\equiv k^m+1$ (mod 9). Then $k^n-k^m=k^m(k^{n-m}-1)$ is divisible by 9. Since the two factors are relatively prime, $10>k$ and $9>k^{n-m}-1$, we can only have k = 3, 6 or 9.

Now $3^3+1=28$ and $3^4+1=82$ show k = 3 is an answer. The case k = 6 cannot work as numbers of the form 6^i+1 end in 7 so that both k^m+1 and k^n+1 would begin and end with 7, which makes $k^n/k^m\geq k$ impossible. Finally, the case k = 9 also cannot work as numbers of the form 9^i+1 end in 0 or 2 so that both numbers would begin and end with 2, which again makes $k^n/k^m\geq k$ impossible.

Other commended solvers: **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6).

Olympiad Corner

(continued from page 1)

Problem 4. Determine all real valued functions $f(x)$ in one real variable for which

$$f\left(f(x)^2+f(y)\right)=xf(x)+y$$

holds for all real numbers x and y.

Problem 5. Determine all integers m for which all solutions of the equation

$$3x^3-3x^2+m=0$$

are rational.

Problem 6. We are given a semicircle with diameter AB. Points C and D are marked on the semicircle, such that AC = CD holds. The tangent of the semicircle in C and the line joining B and D interect in a point E, and the line joining A and E intersects the semicircle in a point F. Show that $CF<FD$ must hold.

對數表的構造

（續第二頁）

這個和確實數值

$$\ln 3=1.09861228866811\ldots$$

相比其精確度也到達第 10 位小數。 讀者不妨自行試算 ln 5, ln 7 等等的數值，然後再和計算機所得的作一比較。

回看上述極為巧妙的計算方法，真的令人佩服當年的數學家們對於數字關係和公式運算的那種創意與觸覺！

【參考文獻】:

項武義教授分析學講座筆記第三章

http://ihome.ust.hk/~malung/391.html

Pell's Equation (II)

(continued from page 2)

The powers are $1+\sqrt{5}$, $6+2\sqrt{5}$, $16+8\sqrt{5}=8(2+\sqrt{5})$, $56+24\sqrt{5}$, $176+80\sqrt{5}=16(11+5\sqrt{5})$, … . Thus, the primitive positive solutions are (x, y) with $x+y\sqrt{5}=2\left(\frac{1+\sqrt{5}}{2}\right)^{6n-5}$ or $2\left(\frac{1+\sqrt{5}}{2}\right)^{6n-1}$. The nonprimitive positive solutions are (x, y) with $x+y\sqrt{5}=2\left(\frac{1+\sqrt{5}}{2}\right)^{6n-3}$. So the general positive solutions are (x, y) with

$$x+y\sqrt{5}=2\left(\frac{1+\sqrt{5}}{2}\right)^k \text{ for odd } k.$$

Then

$$y=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k-\left(\frac{1-\sqrt{5}}{2}\right)^k\right)=F_k,$$

where F_k is the k-th term of the famous *Fibonacci sequence*. Finally, $y^2\equiv 1$ (mod 19) and k should be odd. The smallest such $y=F_{17}=1597$, which leads to $n=\left(F_{17}^2-1\right)/19=134232$.

Comments: For the readers not familiar with the Fibonacci sequence, it is defined by $F_1=1$, $F_2=1$ and $F_{n+1}=F_n+F_{n-1}$ for $n>1$. By math induction, we can check that they satisfy *Binet's formula* $F_n=\left(r_1^n-r_2^n\right)/\sqrt{5}$, where $r_1=(1+\sqrt{5})/2$ and $r_2=(1-\sqrt{5})/2$ are the roots of the *characteristic equation* $x^2=x+1$. (Check cases n = 1, 2 and in the induction step, just use $r_i^{n+1}=r_i^n+r_i^{n-1}$.)