Mathematical Excalibur

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Olympiad Corner

Following are the problems of 2005 Chinese Mathematical Olympiad.

Problem 1. Let $\theta_i \in (-\pi/2, \pi/2)$, i = 1, 2, 3, 4. Prove that there exists $x \in \mathbb{R}$ satisfying the two inequalities

$$\cos^2\theta_1\cos^2\theta_2 - (\sin\theta_1\sin\theta_2 - x)^2 \geq 0$$
$$\cos^2\theta_3\cos^2\theta_4 - (\sin\theta_3\sin\theta_4 - x)^2 \geq 0$$

if and only if

$$\sum_{i=1}^{4}\sin^2\theta_i \leq 2\left(1+\prod_{i=1}^{4}\sin\theta_i+\prod_{i=1}^{4}\cos\theta_i\right).$$

Problem 2. A circle meets the three sides BC, CA, AB of triangle ABC at points D_1, D_2; E_1, E_2 and F_1, F_2 in turn. The line segments D_1E_1 and D_2F_2 intersect at point L, line segments E_1F_1 and E_2D_2 intersect at point M, line segments F_1D_1 and F_2E_2 intersect at point N. Prove that the three lines AL, BM and CN are concurrent.

Problem 3. As in the figure, a pond is divided into $2n$ ($n \geq 5$) parts. Two parts are called neighbors if they have a common side or arc. Thus every part has

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Dept. of Math., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 10, 2005***.

For individual subscription for the next issue, please send us a stamped self-addressed envelope. Send all correspondence to:

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例析數學競賽中的計數問題(三)

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例 12 是否可能將正整數 1，2，3，…，64 分別填入 8×8 的正方形的 64 個小方格內，使得形如圖 1（方向可以任意轉置）的任意四個小方格內數之和總能被 5 整除？試說明理由。

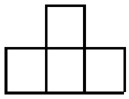

圖 1

解答 不可能。下面用反證法證明：假設圖 2 中 $a,b,c,\cdots,k,l,\cdots$ 就是符合題設填好的數。

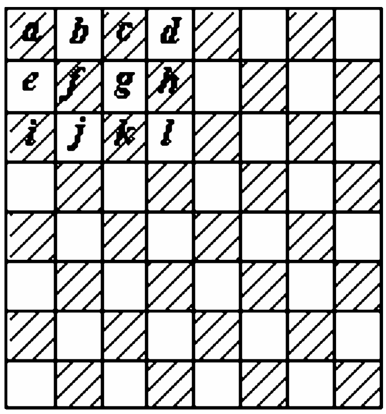

圖 2

因 $5\,|\,b+e+f+g$，$5\,|\,j+e+f+g$，作差有 $5\,|\,b-j$，即 $b \equiv j (\bmod 5)$，設餘數為 r。

同理因 $5\,|\,j+f+b+g$，$5\,|\,e+f+b+g$，故 $5\,|\,j-e$，即 $j \equiv e (\bmod 5)$，顯然其餘數也為 r。

將圖中 64 個小方格染成黑白相間的形式，可得 $b,j,e,g,d,l,\cdots$ 即除角上兩白格中的兩數外，其餘白格中的 $\frac{64}{2}-2=30$ 個數被 5 除都同餘 r。

另一方面，由抽屜原理，1～64 這 64 個正整數中最多有 13 個數被 5 除同餘，與前面得出的結論矛盾！因此，不存在滿足題設的填法。

例 13 平面上給定五點 A、B、C、D、E，其中任何三點不在一直線上。試證：任意地用線段連結某些點（這些線段稱為邊），若所得到的圖形中不出現以這五點中的任何三點為頂點的三角形，則這個圖形不可能有 7 條或更多條邊。

證法 1 （反證法）假設圖形有 7 條或更多條邊，則各點度數和至少是 14。

(1) 若某點度數是 4，則其餘點的度數和至少是 10，由抽屜原理知其中必有一點度數至少是 $\left[\frac{10}{4}\right]+1=3$（度數是 2 就已足夠），故此時必然出現三角形。

(2) 若每點度數至多是 3，由抽屜原理知至少有 4 點的度數是 3，選其中 2 點，不妨設為 A、B，且 A 與 B、C、D 有連線，此時考慮 B 與 A 已有連線，由抽屜原理知 B 必與 C、D 中某一點有連線，這樣也出現了三角形。

而(1)、(2)所得結論都與題設“圖形中不出現以這五點中的任何三點為頂點的三角形”相矛盾，故原命題成立。

證法 2 （反證法）假設圖形有 7 條或更多條邊。

首先我們構造抽屜：每個抽屜裏有三個相異點，共可得 $C_5^3=10$ 個抽屜[注]，又由於同一條邊會在 $5-2=3$ 個抽屜裏出現，則 10 個抽屜裏共有 $7\times3=21$ 條或更多條邊。

由抽屜原理知，至少有一個抽屜裏有 3 條邊，而每條邊在一個三角形中最多出現一次。這 3 條邊恰好與其中不

共線的相異三點構成一個三角形。而這與題設"圖形中不出現以這五點中的任何三點為頂點的三角形"相矛盾，故原命題成立。

注 對於低年級學生計算構造抽屜的個數，我們可以考慮從A、B、C、D、E五點中任取的三個點與剩下的兩點一一對應，而選擇兩點的情形有：AB、AC、AD、AE、BC、BD、BE、CD、CE、DE，共 10 種。

對這個問題稍作引伸，便得下面的問題：

例 14 平面上給定$n(n>3)$個點，其中任何三點不共線。任意地用線段連結某些點 (這些線段稱為邊)，得到x條邊。

(1) 若確保圖形中出現以給定點為頂點的三角形，求證：

$$x \geq \frac{n(n-1)(n-2)+3}{3(n-2)}。$$

當$\frac{n(n-1)(n-2)+3}{3(n-2)}$是整數時，求所有$n$連值及對應$x$的最小值；

(2) 若確保圖形中出現以給定點為頂點的m $(m<n)$ 階完全圖(即m點中任何兩點都有邊連接的圖)，求證：

$$x \geq \frac{C_n^m(C_m^2-1)+1}{C_{n-2}^{m-2}}。$$

證明 (1) Ⅰ・構造抽屜：每個抽屜裏有三個相異點，共可得C_n^3個抽屜。又由於同一條邊會在C_{n-2}^1個抽屜裏出現，根據抽屜原理知，當$x\cdot C_{n-2}^1 \geq 2C_n^3+1$時，才能確保有一個抽屜裏有 3 條邊，而這 3 條邊恰好與其中不共線的相異三點構成一個三角形。

這就是說，確保圖形中出現以給定點為頂點的三角形，則$x \geq \frac{2C_n^3+1}{C_{n-2}^1}$，即

$$x \geq \frac{n(n-1)(n-2)+3}{3(n-2)}。$$

Ⅱ・顯然n，$n-1$，$n-2$中有且只有一個是 3 的倍數。

(i) 當n或$n-1$是 3 的倍數時，一方面

$$\frac{n(n-1)(n-2)+3}{3(n-2)} = \frac{n(n-1)}{3} + \frac{1}{n-2}$$

是整數，則$\frac{1}{n-2}$是整數；另一方面

$n>3$，$n-2>1$，則$\frac{1}{n-2}$是分數。矛盾！此時n無解。

(ii) 當$n-2$是 3 的倍數時，不妨設$n-2=3k$，考慮

$$\begin{aligned}&\frac{n(n-1)(n-2)+3}{3(n-2)}\\ &= \frac{3k(3k+1)(3k+2)+3}{3\cdot 3k}\\ &= \frac{3k(3k^2+3k+1)+1-k}{3k}\\ &= 3k^2+3k+1+\frac{1-k}{3k}\end{aligned}$$

是整數，則$\frac{1-k}{3k}$是整數。

令$\frac{1-k}{3k}=t$，則$k(3t+1)=1$。從而$k=1$，$3t+1=1$即$k=1$，$t=0$。因此

$$n=3k+2=5。$$

從而$x \geq 3k^2+3k+1$，即$x \geq 7$。因此x的最小值是 7。

綜合(i)、(ii)可知，當

$$\frac{n(n-1)(n-2)+3}{3(n-2)}$$

是整數時，$n=5$，$x_{\min}=7$。

(2) 構造抽屜：每個抽屜裏有m個相異點，共可得C_n^m個抽屜。又由於同一條邊會在C_{n-2}^{m-2}個抽屜裏出現。根據抽屜原理知，當

$$x\cdot C_{n-2}^{m-2} \geq C_n^m(C_m^2-1)+1$$

時，才能確保有一個抽屜裏有C_m^2條邊，而這C_m^2條邊恰好與其中不共線的相異m點構成一個m階完全圖。

這就是說，確保圖形中出現以給定點為頂點的m階完全圖，則

$$x \geq \frac{C_n^m(C_m^2-1)+1}{C_{n-2}^{m-2}}。$$

注 題中字母k、m、n、t、x都是指整數。

以上解決數學競賽題的思路與方法告訴我們：見多識廣，可以增強領悟能力；博採眾長，才能減少盲目性。解題中的靈感突現，源自平時的日積月累。只有多鑽研，多探索，做題時便能隨機應變，亦或獨闢蹊徑，以致迎刃而解。

你覺得"數學好玩"嗎？只要你有興趣，數學就會變得迥然不同。你就會感受到數學無盡的魅力，就會具有攻無不克的意志力，就會產生無堅不摧的戰鬥力。如果你根本就沒愛上數學，又怎麼可能碰撞出最為絢爛的火花呢？哪怕非常短暫，瞬間即逝。

有很多同學熱愛數學，都為能在數學奧林匹克的賽場上一試身手、摘金奪銀而默默鑽研，苦苦奮鬥。我想學習中保持長久的數學興趣和培養創造性的思維是成功的關鍵，也是將來可持續發展的保障。而汲取眾家之長是創造性思維的源泉，學會獨立思考是提高創造性思維能力的良策。

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***August 10, 2005.***

Problem 226. Let $z_1, z_2, \ldots, z_n$ be complex numbers satisfying

$$|z_1| + |z_2| + \cdots + |z_n| = 1.$$

Prove that there is a nonempty subset of $\{z_1, z_2, \ldots, z_n\}$ the sum of whose elements has modulus at least 1/4.

Problem 227. For every integer $n \geq 6$, prove that

$$\sum_{k=1}^{n-1} \frac{n}{n-k} \cdot \frac{1}{2^{k-1}} \leq \frac{16}{5}.$$

Problem 228. In $\triangle ABC$, M is the foot of the perpendicular from A to the angle bisector of $\angle BCA$. N and L are respectively the feet of perpendiculars from A and C to the bisector of $\angle ABC$. Let F be the intersection of lines MN and AC. Let E be the intersection of lines BF and CL. Let D be the intersection of lines BL and AC.

Prove that lines DE and MN are parallel.

Problem 229. For integer $n \geq 2$, let a_1, a_2, a_3, a_4 be integers satisfying the following two conditions:

(1) for i = 1, 2, 3, 4, the greatest common divisor of n and a_i is 1 and
(2) for every $k = 1, 2, \ldots, n-1$, we have

$$(ka_1)_n + (ka_2)_n + (ka_3)_n + (ka_4)_n = 2n,$$

where $(a)_n$ denotes the remainder when a is divided by n.

Prove that $(a_1)_n, (a_2)_n, (a_3)_n, (a_4)_n$ can be divided into two pairs, each pair having sum equals n.
(*Source: 1992 Japanese Math Olympiad*)

Problem 230. Let k be a positive integer. On the two sides of a river, there are in total at least 3 cities. From each of these cities, there are exactly k routes, each connecting the city to a distinct city on the other side of the river. Via these routes, people in every city can reach any one of the other cities.

Prove that if any one route is removed, people in every city can still reach any one of the other cities via the remaining routes.
(*Source: 1996 Iranian Math Olympiad, Round 2*)

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Solutions

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Due to an editorial mistake in the last issue, solutions to problems 216, 217, 218, 219 by **D. Kipp Johnson** (teacher, Valley Catholic School, Beaverton, Oregon, USA) were overlooked and his name was not listed among the solvers. We express our apology to him.

Problem 221. (*Due to Alfred Eckstein, Arad, Romania*) The Fibonacci sequence is defined by $F_0 = 1$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Prove that $7F_{n+2}^3 - F_n^3 - F_{n+1}^3$ is divisible by F_{n+3}.

Solution. **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania) and **Kin-Chit O** (STFA Cheng Yu Tung Secondary School).

As $a = 7F_{n+2}^3 + 7F_{n+1}^3$ is divisible by $F_{n+2} + F_{n+1} = F_{n+3}$ and $b = 8F_{n+1}^3 + F_n^3$ is divisible by $2F_{n+1} + F_n = F_{n+2} + F_{n+1} = F_{n+3}$, so $7F_{n+2}^3 - F_n^3 - F_{n+1}^3 = a - b$ is divisible by F_{n+3}.

Other commended solvers: **CHAN Pak Woon** (Wah Yan College, Kowloon, Form 7), **CHAN Tsz Lung**, **CHAN Yee Ling** (Carmel Divine Grace Foundation Secondary School, Form 6), **G.R.A. 20 Math Problem Group** (Roma, Italy), **MA Hoi Sang** (Shun Lee Catholic Secondary School, Form 5), **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6), **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 6) and **WONG Kwok Kit** (Carmel Divine Grace Foundation Secondary School, Form 6).

Problem 222. All vertices of a convex quadrilateral $ABCD$ lie on a circle ω. The rays AD, BC intersect in point K and the rays AB, DC intersect in point L.

Prove that the circumcircle of triangle AKL is tangent to ω if and only if the circumcircle of triangle CKL is tangent to ω.
(*Source: 2001-2002 Estonian Math Olympiad, Final Round*)

Solution. **LEE Kai Seng** (HKUST) and **MA Hoi Sang** (Shun Lee Catholic Secondary School, Form 5).

Let ω_1 and ω_2 be the circumcircles of ΔAKL and ΔCKL respectively. For a point P on a circle Ω, let $\Omega(P)$ denote the tangent line to Ω at P.

Pick D' on $\omega(A)$ so that D and D' are on opposite sides of line BL and pick L' on $\omega_1(A)$ so that L and L' are on opposite sides of line BL.

Next, pick D'' on $\omega(C)$ so that D and D'' are on opposite sides of line BK and pick L'' on $\omega_2(C)$ so that L and L'' are on opposite sides of line BK. Now ω, ω_1 both contain A and ω, ω_2 both contain C. So

$$\begin{aligned} & \omega(A) = \omega_1(A) \\ \Leftrightarrow\ & \angle D'AB = \angle L'AB \\ \Leftrightarrow\ & \angle ADB = \angle ALB \\ \Leftrightarrow\ & BD \parallel LK \\ \Leftrightarrow\ & \angle BDC = \angle KLC \\ \Leftrightarrow\ & \angle BCD'' = \angle KCL'' \\ \Leftrightarrow\ & \omega(C) = \omega_2(C). \end{aligned}$$

Other commended solvers: **CHAN Tsz Lung** and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6).

Problem 223. Let $n \geq 3$ be an integer and x be a real number such that the numbers x, x^2 and x^n have the same fractional parts. Prove that x is an integer.
(*Source: 1997 Romanian Math Olympiad, Final Round*)

Solution. **G.R.A. 20 Math Problem Group** (Roma, Italy).

By hypotheses, there are integers a, b such that $x^2 = x + a$ and $x^n = x + b$. Since x is real, the discriminant $\Delta = 1 + 4a$ of $x^2 - x - a = 0$ is nonnegative. So $a \geq 0$. If $a = 0$, then $x = 0$ or 1.

If $a > 0$, then define integers c_j, d_j so that $x^j = c_j x + d_j$ for $j \geq 2$ by $c_2 = 1$, $d_2 = a > 0$,

$$x^3 = x^2 + ax = (1 + a)x + a$$

leads to $c_3 = 1 + a$, $d_3 = a$ and for $j > 3$, $x^j = (x + a)x^{j-2} = (c_{j-1} + ac_{j-2})x + (d_{j-1} +$

ad_{j-2}) leads to $c_j = c_{j-1} + ac_{j-2} > c_{j-1} > 1$ and $d_j = d_{j-1} + ad_{j-2}$.

Now $c_n x + d_n = x^n = x + b$ with $c_n > 1$ implies $x = (b - d_n)/(c_n - 1)$ is rational. This along with a being an integer and $x^2 - x - a = 0$ imply x is an integer.

Other commended solvers: **CHAN Tsz Lung**, **MA Hoi Sang** (Shun Lee Catholic Secondary School, Form 5), and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6).

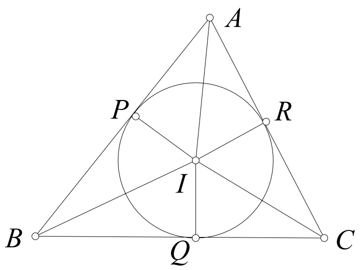
Problem 224. (*Due to Abderrahim Ouardini*) Let a, b, c be the sides of triangle ABC and I be the incenter of the triangle.

Prove that

$$IA \cdot IB \cdot IC \le \frac{abc}{3\sqrt{3}}$$

and determine when equality occurs.

Solution. **CHAN Tsz Lung** and **Kin-Chit O** (STFA Cheng Yu Tung Secondary School).



Let r be the radius of the incircle and s be the semiperimeter $(a + b + c)/2$. The area of $\triangle ABC$ is $(a + b + c)r/2 = sr$ and $\sqrt{s(s-a)(s-b)(s-c)}$ by Heron's formula. So

$$r^2 = (s-a)(s-b)(s-c)/s. \qquad (*)$$

Let P, Q, R be the feet of perpendiculars from I to AB, BC, CA. Now $s = AP + BQ + CR = AP + BC$, so $AP = s-a$. Similarly, $BQ = s-b$ and $CR = s-c$. By the *AM-GM* inequality,

$$s/3 = [(s-a)+(s-b)+(s-c)]/3 \ge \sqrt[3]{(s-a)(s-b)(s-c)}. \qquad (**)$$

Using Pythagoras' theorem, (*) and (**), we have

$$\begin{aligned} &IA^2 \cdot IB^2 \cdot IC^2 \\ &= [r^2+(s-a)^2][r^2+(s-b)^2][r^2+(s-c)^2] \\ &= [(s-a)bc/s][(s-b)ca/s][(s-c)ab/s] \\ &\le (abc)^2/3^3 \end{aligned}$$

with equality if and only if $a = b = c$. The result follows.

Other commended solvers: **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania), **KWOK Lo Yan** (Carmel Divine Grace Foundation Secondary School, Form 5), **MA Hoi Sang** (Shun Lee Catholic Secondary School, Form 5) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6).

Problem 225. A luminous point is in space. Is it possible to prevent its luminosity with a finite number of disjoint spheres of the same size?
(*Source: 2003-2004 Iranian Math Olympiad, Second Round*)

Official Solution.

Let the luminous point be at the origin. Consider all spheres of radius $r = \sqrt{2}/4$ centered at (i, j, k), where i, j, k are integers (not all zero) and $|i|$, $|j|$, $|k| \le 64$. The spheres are disjoint as the radii are less than 1/2. For any line L through the origin, by the symmetries of the spheres, we may assume L has equations of the form $y = ax$ and $z = bx$ with $|a|$, $|b| \le 1$. It suffices to show L intersects one of the spheres.

We claim that for every positive integer n and every real number c with $|c| \le 1$, there exists a positive integer $m \le n$ such that $|\{mc\}| < 1/n$, where $\{x\} = x - [x]$ is the fractional part of x.

To see this, partition [0,1) into n intervals of length $1/n$. If one of $\{c\}$, $\{2c\}$, …, $\{nc\}$ is in $[0,1/n)$, then the claim is true. Otherwise, by the pigeonhole principle, there are $0 < m' < m'' \le n$ such that $\{m'c\}$ and $\{m''c\}$ are in the same interval. Then $|\{m'c\}-\{m''c\}| < 1/n$ implies $|\{mc\}| < 1/n$ for $m = m'' - m' \le n$.

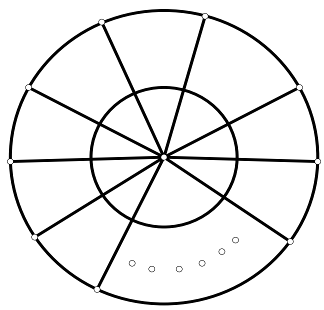
Since $|a| \le 1$, by the claim, there is a positive integer $m \le 16$ such that $|\{ma\}| < 1/16$ and there is a positive integer $n \le 4$ such that $|\{nmb\}| < 1/4$. Now $|\{ma\}| < 1/16$ and $n \le 4$ imply $|\{nma\}| < 1/4$. Then $i = nm \le 64$ and $j = [nma]$, $k = [nmb]$ satisfy $|j-nma| < 1/4$ and $|k-nmb| < 1/4$. So the distance between the point (i, ia, ib) on L and the center (i, j, k) is less than r. Therefore, every line L through the origin will intersect some sphere.

Olympiad Corner

(continued from page 1)

three neighbors. Now there are $4n + 1$ frogs at the pond. If there are three or more frogs at one part, then three of the frogs of the part will jump to the three neighbors respectively.

Prove that at some time later, the frogs at the pond will be uniformly distributed. That is, for any part, either there is at least one frog at the part or there is at least one frog at each of its neighbors.



Problem 4. Given a sequence $\{a_n\}$ satisfying $a_1 = 21/16$ and $2a_n - 3a_{n-1} = 3/2^{n+1}$, $n \ge 2$. Let m be a positive integer, $m \ge 2$.

Prove that if $n \le m$, then

$$\left(a_n + \frac{3}{2^{n+3}}\right)^{1/m}\left(m - \left(\frac{2}{3}\right)^{n(m-1)/m}\right) < \frac{m^2 - 1}{m - n + 1}.$$

Problem 5. Inside and including the boundary of a rectangle $ABCD$ with area 1, there are 5 points, no three of which are collinear.

Find (with proof) the least possible number of triangles having vertices among these 5 points with areas not greater than 1/4.

Problem 6. Find (with proof) all nonnegative integral solutions (x, y, z, w) to the equation

$$2^x \cdot 3^y - 5^z \cdot 7^w = 1.$$